Int. J. Solids Structures, 1971, Vol. 7, pp. 63 to 81. Pergamon Press. Printed in Great Britain

# THE OPTIMAL DESIGN OF BEAMS AND FRAMES WITH COMPLIANCE CONSTRAINTS†

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**Abstract**—The optimal design of simple structures composed of a material which satisfies a viscous power law relation is considered. The structure must be designed for minimum weight subject to a specified rate of work of the external loads. It is shown that the optimal design problem can be treated as a structural design problem with the constitutive relation modified by requirements which are necessary and sufficient to establish optimality. Linear elastic and rigid–plastic problems can be recovered from this treatment.

## 1. INTRODUCTION

In a recent paper Marcal and Prager [1] introduced the concept of an associated structure as a device for the determination of the optimal design of rigid–plastic structures under static loads. Since a cost function (whose argument is a property of the cross-section) must be minimized in order to determine the optimum design, Marcal and Prager conceived an associated structure of identical geometry whose complementary energy density was equal to that of the cost function per unit length of the original structure. Minimization of cost is then converted to a problem of minimization of complementary energy, and hence the optimal design problem is converted into a standard structural problem in the associated structure with a material which is non-linear but reversible. This concept was applied to circular plates by Marcal [2] and further generalized by Prager and Shield [3].

Also in recent years a great deal of work has been devoted to the optimal design of elastic structures for a given compliance under specified static loads. This work is exemplified by the paper of Prager and Taylor [4]. Other references may be found in the comprehensive review paper by Sheu and Prager [5]. In this work it has been usual to develop a kinematic optimality criterion. The optimal displacement field is then determined, and this provides sufficient information to determine the stress field and finally the required cross section properties. These methods have been extended to steady state creep problems by Prager [6].

In this paper we shall reformulate optimal design criteria in order to show that the problems described above can be reduced to a standard structural analysis problem (i.e. one in which generalized stresses and generalized strains must be determined for given loading, support conditions and generalized stress-generalized strain relation). This is effected by using the optimality criterion to eliminate the unknown cross-section properties from the generalized stress-generalized strain relation. For simplicity we shall consider only beams and frames which lie in one plane and are subjected to loads in that plane. We shall further

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assume that bending predominates, and hence that the generalized strains associated with shear and axial forces can be neglected.

Thus for a linear elastic structure, for example, the relation between bending moment $M$ and the associated change in curvature $\kappa$ is given by

$$\kappa = \frac{M}{EI} \tag{1a}$$

where $EI$ is the flexural rigidity. The flexural rigidity is a function of the cross section, and will vary with the single space parameter $s$. We shall find it more convenient to write this linear relation in the form

$$\frac{\kappa}{\kappa_0} = \frac{M}{M_0}, \qquad EI = \frac{M_0}{\kappa_0} \tag{1b}$$

where $\kappa_0$ has the dimensions of $\kappa$ and is constant (i.e. independent of $s$) and $M_0$ has the dimensions of $M$.

Furthermore, we note that the optimal design problem for a linear elastic material defined by equation (1b) and a linear viscous material represented by

$$\frac{\dot{\kappa}}{\dot{\kappa}_0} = \frac{M}{M_0} \tag{1c}$$

(where $\dot{\kappa}_0$ has the dimensions of $\dot{\kappa}$) are identical if we simply interpret displacements as displacement rates and curvature changes as curvature rates. Thus the linear material we consider will be of the form of equation (1c) rather than equation (1b).

Equation (1c) is a special form of the relation often used for steady state creep;

$$\frac{\dot{\kappa}}{\dot{\kappa}_0} = \left(\frac{M}{M_0}\right)^n \tag{2}$$

where $n$ is a positive odd integer. We shall treat such materials in this paper. This relation is such that, as well as giving linear viscous materials when $n = 1$, it permits us to recover rigid–plastic materials when $n$ becomes very large. As $n \to \infty$, we see that $\dot{\kappa} = 0$ when $|M| < M_0$, and $\dot{\kappa}$ is unspecified but has the sign of $M$ when $M = M_0$. The form of the relation given in equation (2) is sketched in Fig. 1 for various values of $n$.

The reformulation given in this paper will be seen to give the results of Marcal and Prager when $n \to \infty$, i.e. for the rigid–plastic case. The results can thus be regarded as a generalization of the work of Marcal and Prager, although it must be emphasized that the approach is different. It further shows that the concept of an artificial "associated structure" introduced by Marcal and Prager is unnecessary; the moment–curvature rate relation of the associated structure is simply the moment–curvature relation of the actual structure modified by the information provided by the optimality criterion.

## 2. THE OPTIMAL DESIGN PROBLEM FOR BEAMS AND FRAMES

As mentioned above, we shall for simplicity consider the optimal design of beams and frames which lie in one plane and are loaded in that plane. In such structures a single spatial variable $s$ may be used to locate any point or cross section of the structure. As above, we

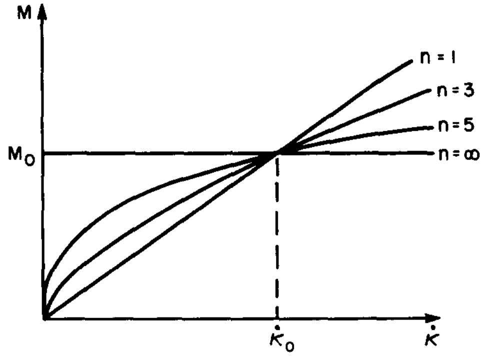


FIG. 1.

assume that bending deformations predominate, and the generalized strain rates associated with shear and axial forces will in consequence be neglected. The curvature rates $\dot{\kappa}(s)$ and the bending moments $M(s)$ are related by equation (2),

$$\frac{\dot{\kappa}}{\dot{\kappa}_0} = \left(\frac{M}{M_0}\right)^n$$

where $n$ is an odd positive integer. For the purpose of the optimal design problem, we assume that $\dot{\kappa}_0$ is greater than zero and constant, i.e. it is not a function of $s$. $M_0$ must be non-negative, and will in general be a function of $s$.

The cost of any cross section of the structure will be assumed to be represented by

$$\phi = \phi(M_0) \tag{3}$$

The cost of the structure is then given by

$$C = \int_S \phi(M_0)\,\mathrm{d}s \tag{4}$$

where $S$ represents integration over the total length of the structure.

The structure is assumed to be supported at a number of points (not less than the number required to prevent rigid body motion) at which the displacement rates or rotation rates are given zero. Generalized forces $\mathbf{P}$ are assumed to act on the structure. $\mathbf{P}$ may be a single point load, a group of point loads, a distributed load, and so on. Generalized displacement rates or velocities $\mathbf{v}$ are defined such that $\mathbf{P}\,.\,\mathbf{v}$ has the dimensions of work. The generalized velocities may then be components of velocity at particular points in particular directions, average velocities or whatever else is dictated by the nature of the loads. It is further required in the problem under consideration that the rate of work done by the external loads has a prescribed value

$$\mathbf{P}\,.\,\mathbf{v} = D \tag{5}$$

The optimal design problem can now be succinctly stated. We are required to find $M_0(s)$ such that the cost of the structure $C$ [equation (4)] is minimized subject to the constraint that the rate of work done by the prescribed loads $\mathbf{P}$ is $D$ [equation (5)].

## 3. NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

In order to determine necessary conditions for optimality by means of variational methods, we introduce a functional

$$J = C + \lambda^2 \mathbf{P} \, . \, \mathbf{v} \tag{6}$$

where $\lambda^2$ is a Lagrangian multiplier. Conditions which ensure that the first variation of $J$ is zero will then be necessary conditions for a stationary value of $C$ subject to the constraint $\mathbf{P} \, . \, \mathbf{v} = D =$ const. To place the constraint in a suitable form we note that for given $M_0(s)$ there will be associated with the loads $\mathbf{P}$ moments $M(s)$ which are statically admissible, and kinematically admissible curvature rates $\dot{\kappa}(s)$ obtained from $M(s)$ by means of the constitutive relation (2). Hence, from a balance of internal and external work rates

$$\begin{aligned} \mathbf{P} \, . \, \mathbf{v} &= \int_S M\dot{\kappa} \, \mathrm{d}s \\ &= \int_S \dot{\kappa}_0 \frac{M^{n+1}}{M_0^n} \, \mathrm{d}s = \int_S M_0 \frac{\dot{\kappa}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}} \, \mathrm{d}s \end{aligned} \tag{7}$$

after using equation (2).

Thus, substituting from equations (4) and (7)

$$J = \int_S \phi(M_0) \, \mathrm{d}s + \lambda^2 \int_S \dot{\kappa}_0 \frac{M^{n+1}}{M_0^n} \, \mathrm{d}s \tag{8}$$

We now vary $M_0$ by $\delta M_0$, noting that in this change $M$ will change to $(M + \delta M)$ if the structure is statically indeterminate. $M_0(s)$ is assumed to be a continuous function in the optimal design. Hence

$$\delta J = \int_S \left\{ \frac{\mathrm{d}\phi}{\mathrm{d}M_0} - n\lambda^2 \dot{\kappa}_0 \left( \frac{M}{M_0} \right)^{n+1} \right\} \delta M_0 \, \mathrm{d}s + (n+1)\lambda^2 \dot{\kappa}_0 \int_S \delta M \left( \frac{M}{M_0} \right)^n \mathrm{d}s. \tag{9}$$

The second integral may be written as

$$\int_S \delta M \dot{\kappa} \, \mathrm{d}s.$$

If the structure is statically determinate $\delta M = 0$. If the structure is statically indeterminate we note that $\delta M$ is in equilibrium with zero external load, $\int_S \delta M \dot{\kappa} \, \mathrm{d}s = 0$ if $\dot{\kappa}(s)$ is kinematically admissible. This we certainly require as a necessary condition in the optimal design. Since $\delta M_0$ is an arbitrary variation, $\delta J = 0$ requires that

$$\frac{\mathrm{d}\phi}{\mathrm{d}M_0} = n\lambda^2 \dot{\kappa}_0 \left( \frac{M}{M_0} \right)^{n+1} = n\lambda^2 \dot{\kappa}_0 \left( \frac{\dot{\kappa}}{\dot{\kappa}_0} \right)^{(n+1)/n}. \tag{10}$$

Thus, given that $M(s)$ is statically admissible, (9) provides two necessary conditions that the cost should be stationary;

(i) $\dot{\kappa}(s)$ must be kinematically admissible,

(ii) equation (10) must be satisfied.

Provided that $\phi(M_0)$ is a *convex function*, we can show that these conditions are also sufficient to ensure a global minimum for $C$. If $\hat{M}_0$ and $\overline{M}_0$ are two independent values, $\phi(M_0)$ is by definition convex if

$$\phi(\hat{M}_0) - \phi(\overline{M}_0) \geq (\hat{M}_0 - \overline{M}_0) \left. \frac{\mathrm{d}\phi}{\mathrm{d}M_0} \right|_{M_0 = \overline{M}_0}. \tag{11}$$

Let $\hat{M}_0(s)$ be any arbitrary design, and $\overline{M}_0(s)$ be a design satisfying the necessary conditions given above. (11) may be integrated over the total length of the structure to give

$$\int_S \phi(\hat{M}_0)\,\mathrm{d}s - \int_S \phi(\overline{M}_0)\,\mathrm{d}s \geq \int_S (\hat{M}_0 - \overline{M}_0)\frac{\mathrm{d}\phi}{\mathrm{d}M_0}\bigg|_{M_0=\overline{M}_0} \mathrm{d}s$$

$$= n\lambda^2 \int_S (\hat{M}_0 - \overline{M}_0)\frac{\dot{\kappa}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}} \tag{12}$$

after using equation (10). By means of the analogue of the potential energy theorem for elasticity (Hoff [7])

$$\int_S \hat{M}_0 \frac{\bar{\dot{\kappa}}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}}\,\mathrm{d}s \geq \int_S \hat{M}_0 \frac{\hat{\dot{\kappa}}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}}\,\mathrm{d}s \tag{13}$$

since $\bar{\dot{\kappa}}$ is kinematically admissible and $\hat{\dot{\kappa}}$ is the correct curvature rate field associated with the $\hat{M}_0$ structure, and both fields are compatible with generalized velocities $\mathbf{v}$. Further

$$\int \hat{M}_0 \frac{\hat{\dot{\kappa}}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}}\,\mathrm{d}s = \int \overline{M}_0 \frac{\bar{\dot{\kappa}}^{(n+1)/n}}{\dot{\kappa}_0^{1/n}}\,\mathrm{d}s = \mathbf{P}\,.\,\mathbf{v} \tag{14}$$

since internal and external work rates must balance in each design [cf. equation (7)]. Substituting (14) and (13) into (12), we see that provided that $\phi(M_0)$ is convex

$$\hat{C} = \int_S \phi(\hat{M}_0)\,\mathrm{d}s \geq \int_S \phi(\overline{M}_0)\,\mathrm{d}s = \overline{C}. \tag{15}$$

We note that $\hat{M}_0$ in the design which does not satisfy the necessary conditions for optimality is not required to be continuous, and hence global optimality is ensured.

## 4. THE DETERMINATION OF THE OPTIMAL SOLUTION

We may now proceed to outline a method for determining the details of the optimal design. In this particular approach, we shall use the optimality condition [equation (10)]

$$\frac{\mathrm{d}\phi}{\mathrm{d}M_0} = n\lambda^2\dot{\kappa}_0\left(\frac{M}{M_0}\right)^{n+1}$$

to eliminate $M_0$ from the constitutive relation for the structure [equation (2)]

$$\frac{\dot{\kappa}}{\dot{\kappa}_0} = \left(\frac{M}{M_0}\right)^n.$$

This can always be done by solving equation (10) for $M_0$ and substituting into equation (2). However, an analytical expression can be found only for certain simple cases. For simplicity, we shall assume that

$$\phi(M_0) = (M_0)^k \tag{16}$$

where $k \geq 1$. $\phi(M_0)$ will then be a convex function. It must be emphasized that the same procedure can equally well be applied to more complex cost functions, although it will generally be necessary to consider particular expressions. From (16), we note that

$$\frac{\mathrm{d}\phi}{\mathrm{d}M_0} = k(M_0)^{k-1} \tag{17a}$$

and

$$M_0\frac{\mathrm{d}\phi}{\mathrm{d}M_0} = k(M_0)^k = k\phi(M_0). \tag{17b}$$

Equation (10) now becomes

$$k(M_0)^{k-1} = n\lambda^2\dot{\kappa}_0\left(\frac{M}{M_0}\right)^{n+1}. \tag{18}$$

Thus

$$(M_0)^{(n+1)+(k-1)} = \frac{n\lambda^2\dot{\kappa}_0(M)^{n+1}}{k} \tag{19}$$

In this equation $M$ is raised to an even power, and hence the right-hand side is positive. To ensure that $M_0$ is positive it is convenient to replace $M$ by $|M|$, and assume that all roots are positive. To further reduce the complexity of the expression, put

$$\alpha = k-1 \geq 0 \tag{20}$$

Equation (19) then gives

$$M_0 = \left(\frac{n\lambda^2\dot{\kappa}_0}{k}\right)^{\frac{1}{n+1+\alpha}}\{|M|\}^{\frac{n+1}{n+1+\alpha}} \tag{21}$$

Equation (21) is now substituted into equation (2) to give

$$\dot{\kappa} = \frac{\dot{\kappa}_0(M)^n}{[(n\lambda^2\dot{\kappa}_0)/k]^{\frac{n}{n+1+\alpha}}\{|M|\}^{\frac{n(n+1)}{n+1+\alpha}}} \tag{22}$$

This relation can be written in the form

$$\dot{\kappa} = \langle M\rangle\frac{(\dot{\kappa}_0)^{\frac{1+\alpha}{n+1+\alpha}}}{(n\lambda^2/k)^{\frac{n}{n+1+\alpha}}}\{|M|\}^{\frac{n\alpha}{n+1+\alpha}} \tag{23}$$

where

$$\langle M\rangle = +1 \text{ if } M > 0,$$
$$\langle M\rangle = -1 \text{ if } M < 0.$$

In this relation $\lambda^2$ is regarded as a constant; it is in fact a scaling factor whose significance will be discussed later. For all values of $k$ and $\alpha$, equation (23) is a relation for which $\mathrm{d}M/\mathrm{d}\dot{\kappa} \geq 0$ for all values of $M$. As an example of the form of this relation, it may be seen that if $\alpha = 0$ ($k = 1$) the exponent of $|M|$ is zero. This relation is then as sketched in Fig. 2.

The optimal design problem has now been reduced to a *standard structural analysis problem.* Loads **P** are prescribed on the structure. If the structure is statically determinate,

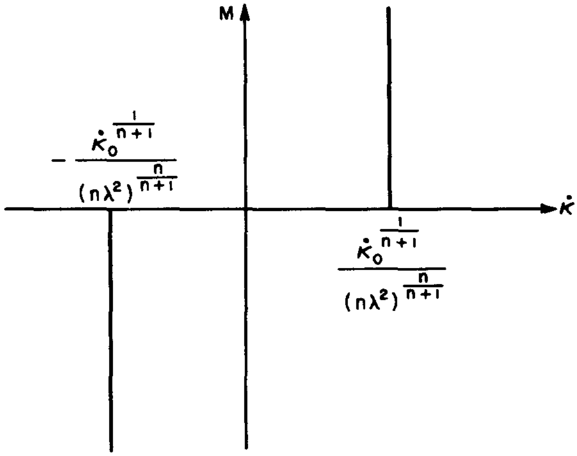


FIG. 2.

$M$ is uniquely determined, as are the curvature rates after using equation (23). If the structure is statically indeterminate, the moment distribution $M(s)$ which leads to curvature rates which are kinematically admissible must be chosen from among the class of statically admissible bending moment distributions.

The analogy between this presentation and the technique described by Marcal and Prager [1] is now apparent, although Marcal and Prager considered only the rigid–plastic case and we have not yet discussed the validity of the results as $n$ tends to infinity. Marcal and Prager introduced an associated structure with a constitutive relation which is a less general form of (23).

The modified complementary energy theorem (Hoff [7]) is probably the most general method of representing the solution to the indeterminate problem. The "complementary energy" density $\Omega(M)$ is obtained from (23)

$$\Omega(M) = \int \dot{\kappa}\, dM = \frac{(n+1)(\alpha+1)}{(n+\alpha+1)} \frac{(\dot{\kappa}_0)^{\frac{1+\alpha}{n+1+\alpha}}}{(n\lambda^2/k)^{\frac{n}{n+1+\alpha}}} |M|^{\frac{(n+1)(\alpha+1)}{(n+1+\alpha)}} \tag{24}$$

The statically admissible bending moment distribution which leads to a kinematically admissible velocity distribution is that which minimizes

$$\int_S \Omega(M)\, ds \quad \text{or} \quad \int_S |M|^{\frac{(n+1)(\alpha+1)}{n+1+\alpha}}\, ds$$

since the coefficients in equation (24) are constants. The stability of equation (23) (i.e. $dM/d\dot{\kappa} \geq 0$) ensures that there is a unique solution to this problem.

The velocity constraint, in the form $\mathbf{P}\,.\,\mathbf{v} = D$, has not yet been introduced. This is satisfied by an appropriate choice of $\lambda^2$. The work rate balance gives

$$\begin{aligned} D = \mathbf{P}\,.\,\mathbf{v} &= \int_S M\dot{\kappa}\, ds \\ &= \frac{\dot{\kappa}_0^{\frac{1+\alpha}{n+1+\alpha}}}{(n\lambda^2/k)^{\frac{n}{n+1+\alpha}}} \int |M|^{\frac{(n+1)(\alpha+1)}{n+1+\alpha}}\, ds. \end{aligned} \tag{25}$$

Solving for $\lambda^2$, we obtain

$$\lambda^2 = \frac{k\dot{\kappa}_0^{\frac{1+\alpha}{n}}}{nD^{\frac{n+1+\alpha}{n}}}\left[\int_S |M|^{\frac{(n+1)(\alpha+1)}{(n+1+\alpha)}}\,\mathrm{d}s\right]^{\frac{n+1+\alpha}{n}}. \tag{26}$$

In statically indeterminate problems, the minimum value of the integral in square brackets must be used; this presents no difficulties, since the minimum value of $\int_S \Omega(M)\,\mathrm{d}s$ can be found without prior knowledge of $\lambda^2$.

Once $M(s)$ has been determined, $M_0(s)$ can be found by substituting $M(s)$ into equation (21). Alternatively, $M_0(s)$ can be obtained by substituting $M(s)$ and $\dot{\kappa}(s)$ found from equation (23) into the constitutive equation (2). This alternative method is not computationally easier, but it does illustrate physically the process by which the real material behavior is regained from the constitutive equation (23). Consider again the case $k = 1$, $\alpha = 0$ for which the $M-\dot{\kappa}$ relation corresponding to (23) is shown in Fig. 2. Suppose at some particular point in the structure the bending moment is found to be $M(s^*)$. A point $F$ may then be plotted on the $M$–$\dot{\kappa}$ diagram (Fig. 3) which represents the moment and curvature at section $s^*$. We must then determine a value of $M_0 = M_0(s^*)$ such that the curve given by equation (2) passes through point $F$.

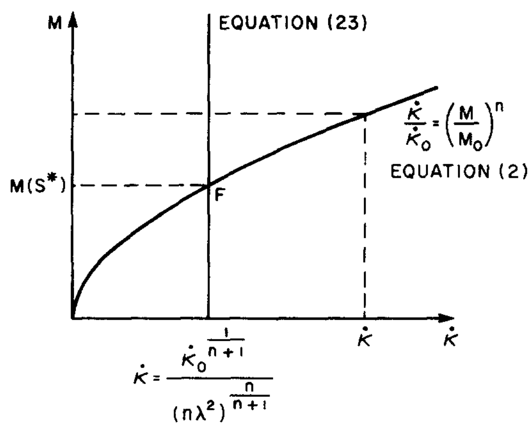


FIG. 3.

The cost of the structure may be determined directly by multiplying equation (18) by $M_0$ and integrating over the structure. Using equations (17b), (7) and (26)

$$C = \int_S \phi\,\mathrm{d}s = \int_S (M_0)^{1/k}\,\mathrm{d}s = \frac{n\lambda^2}{k}\int_S \dot{\kappa}_0 \frac{M^{n+1}}{M_0^n}\,\mathrm{d}s$$

$$= \left(\frac{n\lambda^2}{k}\right)\mathbf{P}\,.\,\mathbf{v}$$

$$= \left(\frac{\dot{\kappa}_0}{D}\right)^{\frac{1+\alpha}{n}}\left[\int_S |M|^{\frac{(n+1)(1+\alpha)}{n+1+\alpha}}\,\mathrm{d}s\right]^{\frac{n+1+\alpha}{n}} \tag{27}$$

The value of $\dot{\kappa}_0$ which is used in the constitutive equation (2) is itself an arbitrary choice. Under certain circumstances it may be convenient to choose a particular value of $\dot{\kappa}_0$ given by

$$\dot{\kappa}_0^* = \frac{k}{n\lambda^2} \tag{28}$$

which results in a simplified version of many of the equations. The appropriate value of $\dot{\kappa}_0$ for which (28) holds may be obtained by substituting (28) into (26) to give

$$\dot{\kappa}_0^* = \frac{D}{\left[\int_S |M|^{\frac{(n+1)(\alpha+1)}{(n+1+\alpha)}} \,\mathrm{d}s\right]} \tag{29}$$

In the case of an indeterminate structure, the integral in equation (29) is again the minimum value for $M(s)$ statically admissible. This integral can be evaluated without prior knowledge of $\dot{\kappa}_0$ or any of the other parameters determined by the optimal design.

If the substitutions of equations (28) and (29) are made, the optimality condition (18) becomes

$$(M_0)^\alpha = \left(\frac{M}{M_0}\right)^{n+1} = \left(\frac{\kappa}{\dot{\kappa}^*}\right)^{\frac{n+1}{n}}. \tag{30}$$

The expression for $M_0$ [equation (21)] in turn reduces to

$$M_0 = \{|M|\}^{\frac{n+1}{n+1+\alpha}} \tag{31}$$

The constitutive equation, after elimination of $M_0$ [equation (23)] is now

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \langle M\rangle\{|M|\}^{\frac{n\alpha}{n+1+\alpha}} \tag{32}$$

where $\langle M\rangle$ is defined as before. The cost of the optimal structure [equation (27)] takes the simple form

$$C = \left[\int |M|^{\frac{(n+1)(1+\alpha)}{n+1+\alpha}} \mathrm{d}s\right]. \tag{33}$$

It should be noted that for the linear cost relation $\phi = M_0(\alpha = 0)$ the most important features of the optimal design are independent of $n$, especially if the substitution of equations (28) and (29) is used.

## 5. RIGID–PLASTIC STRUCTURES

As we pointed out in the introductory section, the constitutive equation (2) represents a rigid–plastic material if $n$ becomes infinitely large. We wish to examine now whether the problem defined in Section 2, the necessary and sufficient conditions given in Section 3 and the methods for determining the optimal design given in Section 4 do indeed apply to rigid plastic structures as $n$ tends to infinity. In this limiting case $M_0$ must be interpreted as the limit moment, and it is appropriate to express the cost $C$ as a function of $M_0$ as in equation (4).

When the loads on a rigid–plastic structure are increased monotonically, the structure will remain rigid until the limit value of the load parameter is reached. Thereupon flow will occur at constant load. The problem formulated in Section 2 is thus perfectly appropriate for the limiting case when $n$ is infinitely large; by requiring that the rate of work done by the loads is $D$ [equation (5)] we imply that the structure will flow under loads $\mathbf{P}$. We must recognize that the particular value of $D$, however, is not important since all values of $D > 0$ can be achieved when flow takes place.

The methods used for determining the optimal solution in Section 4 were based on the necessary conditions for optimality and the particular convex cost function given in equation (16). We shall proceed by examining the equations given in Section 4 for the limiting case $n \to \infty$, and then return to determine whether the conditions are necessary and sufficient for an optimal design.

We note first, from equation (26), that $(n\lambda^2/k)$ has a finite value as $n$ tends to infinity.

$$\frac{n\lambda^2}{k} = \frac{1}{D}\left[\int_S |M|^{\alpha+1}\,\mathrm{d}s\right]. \tag{34}$$

We may then arbitrarily put $\dot{\kappa}_0 = \dot{\kappa}_0^* = k/n\lambda^2$ as in equation (28). Substitution of this into equation (34) specifies the arbitrary positive quantity $D$

$$D = \dot{\kappa}_0^*\left[\int_S |M|^{\alpha+1}\,\mathrm{d}s\right]. \tag{35}$$

Equations (30)–(33) are now applicable. It is clear that, for $n \to \infty$, equation (30) can be satisfied only if

$$|M| = M_0. \tag{36}$$

The substitution for $\dot{\kappa}$ cannot be carried out [as in the second part of equation (30)] since for $|M| = M_0$ and $n \to \infty$, $\dot{\kappa}$ from equation (2) is indeterminate. Equation (36) agrees with equation (32) with $n \to \infty$.

Despite the indeterminacy of $\dot{\kappa}$ in equation (30), we adopt equation (32) with $n \to \infty$. This gives

$$\dot{\kappa} = \dot{\kappa}_0^*\langle M\rangle|M|^{\alpha}. \tag{37}$$

With a relation between $\dot{\kappa}$ and $M$ independent of $M_0$ established, we proceed as before to minimize

$$\int_S \Omega(M)\,\mathrm{d}s = \dot{\kappa}_0^*\int_S |M|^{\alpha+1}\,\mathrm{d}s \tag{38}$$

in order to determine $M(s)$ which, through equation (37), leads to kinematically admissible curvature rates. The determination of $M_0$ from these values of $M(s)$ can still be thought of as fitting a rigid–plastic constitutive equation through the $M$, $\dot{\kappa}$ point for each section, as in Fig. 3. The cost of the rigid–plastic design, from equation (33), is

$$C = \left[\int_S |M|^{\alpha+1}\,\mathrm{d}s\right] \tag{39}$$

where $M(s)$ is that (statically admissible moment distribution) which minimizes $\int_S \Omega(M)\, ds$.

Is the design carried out in this way optimal? The information obtained from equation (30), which was in turn obtained from the variational principle, is that $|M| = M_0$. This, it may be suspected, is the necessary condition for an optimal design. This can easily be established independently (e.g. Heyman [8]) by means of the lower bound theorem of limit analysis. If $|M| < M_0$ at any section, the section is not fully stressed, and less material can be used i.e. $M_0$ can be decreased until it is equal to $|M|$.

In equation (37) we have arbitrarily put

$$|\dot{\kappa}| = \dot{\kappa}_0^*|M|^\alpha = \frac{\dot{\kappa}_0^*}{k}\frac{d\phi}{dM_0}. \tag{40}$$

Return now to equation (12), with $\hat{M}_0$ as any arbitrary design which flows under the given loads $\mathbf{P}$, and $\overline{M}_0$ the design determined by use of equation (36) and (40). Convexity of the cost function gives

$$\int_S \phi(\hat{M}_0)\, ds - \int_S \phi(\overline{M}_0)\, ds \geq \int_S (\hat{M}_0 - \overline{M}_0)\frac{d\phi}{dM_0}\bigg|_{M_0=\overline{M}_0} ds = k\int_S (\hat{M}_0 - \overline{M}_0)\frac{|\dot{\kappa}|}{\dot{\kappa}_0^*}\, ds. \tag{41}$$

By the upper bound theorem of limit analysis

$$\int_S \hat{M}_0 \frac{|\dot{\kappa}|}{\dot{\kappa}_0^*}\, ds \leq \mathbf{P}\,.\,\mathbf{v} = D. \tag{42}$$

Further

$$\int \overline{M}_0 \frac{|\dot{\kappa}|}{\dot{\kappa}_0^*} = \mathbf{P}\,.\,\mathbf{v} = D \tag{43}$$

and hence

$$\hat{C} = \int \phi(\hat{M}_0)\, ds \geq \int \phi(\overline{M}_0)\, ds = \overline{C}. \tag{44}$$

This result holds if

(i) $\dot{\kappa}$ is kinematically admissible,
(ii) $|\dot{\kappa}|$ is proportional to $d\phi/dM_0$ at *every* point in the structure,
(iii) $M_0 = |M|$.

These three conditions are thus sufficient conditions for an optimal design, and they are met in the procedure described in this section.

The conditions for finite $n$ and $n \to \infty$ differ therefore in only one respect; when $n \to \infty$ there are no longer necessary conditions on the curvatures. This is readily explained, since because of the indeterminate curvatures given by the rigid–plastic constitutive relation, the optimality designed structure does not *necessarily* flow with $|\dot{\kappa}|$ proportional to $d\phi/dM_0$, even though this condition forms part of the sufficient conditions for determining the optimal design.

The approach presented here is identical in execution to the method of Marcal and Prager [1] for the rigid–plastic case, even though the formulation is arrived at in a rather different manner.

## 6. OPTIMAL DESIGN WITH A MINIMUM CROSS-SECTION

In many optimal designs derived from the methods discussed above, the value of $M_0$ at certain cross sections is found to be zero. This is an impractical result, in part because such sections may nevertheless be required to transmit shear forces. This impractical feature of the design can be eliminated if it is stipulated that $M_0$ can never be smaller than some specified value, say $M_0^*$. With this additional constraint the results of Section 3 are no longer valid, since $\delta M_0$ in equation (9) is no longer an arbitrary function.

Indeed, it should be pointed out that under the conditions set out in Section 3 $\delta M_0$ is not truly an arbitrary function, since it can only be positive when $M_0 = 0$. If this occurs only at isolated points in the structure it presents no difficulties. However in some optimal designs it can be expected that $M_0$ will be zero over finite regions of the structure. Thus the generalizations presented in this section are in fact applicable to the case where $M_0$ may have any positive value and $M_0^* = 0$.

The changes in the variational principle necessary to deal with a specified minimum value of $M_0$ are straight-forward and will not be given in detail. It is assumed that $M_0$ is continuous, and that in the optimal design $M_0 = M_0^*$ over various regions of the beam denoted collectively by $S_1$. Over the remainder of the beam, $S_2$, $M_0 > M_0^*$. $M_0$ is permitted to vary by an infinitesimal amount in $S_2$, and small variations in the boundaries of the region $S_2$ are also permitted. The requirement that the first variation of $J$ [equation (8)] should be zero leads to a necessary condition that is identical to equation (10) except that it applies only in $S_2$. Thus

$$\frac{\mathrm{d}\phi}{\mathrm{d}M_0} = n\lambda^2\dot{\kappa}_0\left(\frac{M}{M_0}\right)^{n+1} = n\lambda^2\dot{\kappa}_0\left(\frac{\dot{\kappa}}{\dot{\kappa}_0}\right)^{(n+1)/n} \quad \text{in } S_2. \tag{45}$$

In addition, it is necessary that $\dot{\kappa}$ should be kinematically admissible.

In the region $S_1$ the curvature must be given by

$$\frac{\kappa}{\dot{\kappa}_0} = \left(\frac{M}{M_0^*}\right)^n. \tag{46}$$

A design satisfying these conditions can be found. It is useful to consider the conditions at a point on the boundary of $S_1$ and $S_2$. Let the moment and curvature rate at such a point be given by $M^*, \dot{\kappa}^*$ respectively. Because of the continuity of $M_0(s)$ both equations (45) and (46) must apply at this point. From (45) it is evident that

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}M_0}\right|_{M_0=M_0^*} = n\lambda^2\dot{\kappa}_0\left(\frac{\dot{\kappa}^*}{\dot{\kappa}_0}\right)^{(n+1)/n}. \tag{47}$$

We proceed on the assumption that in $S_1$,

$$n\lambda^2\dot{\kappa}_0\left(\frac{\dot{\kappa}}{\dot{\kappa}_0}\right)^{(n+1)/n} = n\lambda^2\dot{\kappa}_0\left(\frac{M}{M_0^*}\right)^{n+1} \leq \left.\frac{\mathrm{d}\phi}{\mathrm{d}M_0}\right|_{M_0=M_0^*} \tag{48}$$

This condition is necessary in order to remove any ambiguity as to whether (45) or (46) applies at any point when $S_1$ and $S_2$ are not known *a priori*, and implies that the maximum values of $|\dot{\kappa}|$ and $|M|$ in any region where $M_0 = M_0^*$ occur at the ends of that region.

We may now proceed to show that equation (45) in $S_2$ and equations (46) and (48) in $S_1$, together with the condition that $\dot{\kappa}(s)$ must be kinematically admissible, are sufficient for global optimality provided that the cost function is convex.

The cost function $\phi(M_0)$ can be chosen to be zero when $M_0 = M_0^*$, since no cost penalty is incurred when the minimum cross section is used. Let $\hat{M}_0(s)$, such that $\hat{M}_0(s) \geq M_0^*$, be any design which satisfies the velocity constraint $\mathbf{P} \cdot \mathbf{v} = D$. Let $\overline{M}_0(s)$ be a design which in addition satisfies the conditions given above. We shall continue to use $S_1$ and $S_2$ to refer to the regions in which $M_0 = M_0^*$ and $M_0 > M_0^*$ respectively in the design which satisfies the necessary conditions for equilibrium. Convexity of the cost function implies that

$$\phi(\hat{M}_0) - \phi(\overline{M}_0) \geq (\hat{M}_0 - \overline{M}_0)\left.\frac{\mathrm{d}\phi}{\mathrm{d}M_0}\right|_{M_0=\overline{M}_0}. \tag{49}$$

Consider first the region $S_2$. Integrate over $S_2$, and substitute for $\mathrm{d}\phi/\mathrm{d}M_0$ from equation (45)

$$\int_{S_2} \phi(\hat{M}_0)\,\mathrm{d}s - \int_{S_2} \phi(\overline{M}_0)\,\mathrm{d}s \geq n\lambda^2\dot{\kappa}_0 \int_{S_2} (\hat{M}_0 - \overline{M}_0)\left(\frac{\dot{\bar{\kappa}}}{\dot{\kappa}_0}\right)^{(n+1)/n} \mathrm{d}s \tag{50}$$

Consider next the region $S_1$. In this region $\overline{M}_0 = M_0^*$, and $\phi(\overline{M}_0)$ is in fact zero. Integrating equation (49) over $S_1$, and using equation (48), we may write

$$\begin{aligned} \int_{S_1} \phi(\hat{M}_0)\,\mathrm{d}s - \int_{S_1} \phi(\overline{M}_0)\,\mathrm{d}s &\geq \int_{S_1} (\hat{M}_0 - \overline{M}_0)\left.\frac{\mathrm{d}\phi}{\mathrm{d}M_0}\right|_{M_0=M_0^*} \\ &\geq n\lambda^2\dot{\kappa}_0 \int_{S_1} (\hat{M}_0 - \overline{M}_0)\left(\frac{\dot{\bar{\kappa}}}{\dot{\kappa}_0}\right)^{(n+1)/n} \mathrm{d}s. \end{aligned} \tag{51}$$

Finally, adding (50) and (51),

$$\int_S \phi(\hat{M}_0)\,\mathrm{d}s - \int_S \phi(\overline{M}_0)\,\mathrm{d}s \geq n\lambda^2\dot{\kappa}_0 \int_S (\hat{M}_0 - M_0)\left(\frac{\dot{\bar{\kappa}}}{\dot{\kappa}_0}\right)^{(n+1)/n} \mathrm{d}s. \tag{52}$$

This expression is identical to equation (12), and by the same reasoning that was used earlier, the right hand side of inequality (52) is itself greater than or equal to zero. Hence

$$\int \phi(\hat{M}_0)\,\mathrm{d}s \geq \int \phi(\overline{M}_0)\,\mathrm{d}s \tag{53}$$

showing that the necessary conditions are also sufficient for global optimality.

In order to determine the optimal design we again use the device of eliminating $M_0$ from the constitutive relations. As a simple example, let

$$\begin{aligned} \phi(M_0) &= M_0^k - (M_0^*)^k \quad \text{for} \quad M_0 \geq M_0^* \\ \phi(M_0) &= 0 \qquad\qquad\qquad\quad\; M_0 < M_0^*. \end{aligned} \tag{54}$$

Equation (45) then gives, with $\alpha = k - 1$,

$$M_0 = \left(\frac{n\lambda^2\dot{\kappa}_0}{k}\right)^{\frac{1}{n+1+\alpha}} \{|M|\}^{\frac{n+1}{n+1+\alpha}} \tag{55}$$

and substitution into equation (2) leads to

$$\frac{\dot{\kappa}}{\dot{\kappa}_0} = \langle M \rangle \frac{1}{\left(\frac{n\lambda^2 \dot{\kappa}_0}{k}\right)^{\frac{n}{n+1+\alpha}}} \{|M|\}^{\frac{n\alpha}{n+1+\alpha}} \tag{56}$$

This equation is identical to equation (23), but it is subject to the restriction of equation (48) and applies only for

$$|M| \geq \left\{ \frac{(M_0^*)^{n+1+\alpha}}{n\lambda^2 \dot{\kappa}_0 / k} \right\}^{1/(n+1)}. \tag{57}$$

For $|M|$ less than the quantity on the right hand side of (57), equation (46) applies. Equations (46) and (57) provide a piecewise continuous moment curvature relation which is independent of $M_0$. The problem is thus again reduced to a structural analysis problem, in that the statically admissible moments $M(s)$ which lead to kinematically admissible curvature rates when (45) and (57) are applied must be found. This constitutive relation is plotted for the case $k = 1$, $\alpha = 0$ in Fig. 4.

The case of a rigid–plastic material ($n \to \infty$) can be dealt with as before and will not be discussed in detail. The regions where $M_0 = M_0^*$ become rigid regions where no deformation occurs. The curvature condition in $S_2$ is no longer a necessary condition, but as pointed out by Marcal and Prager [1] and confirmed by the appropriate form of the equations given in this section, it is a sufficient condition.

Because of the piecewise continuous nature of the moment–curvature relation when a minimum cross section is introduced, it is particularly helpful to use the substitution of equation (28) in carrying out computations. This effectively permits us to compute $\lambda^2$ after the solution has been found rather than before. Thus, putting $\dot{\kappa}_0 = \dot{\kappa}_0^*$, where

$$\dot{\kappa}_0^* = \frac{k}{n\lambda^2}$$

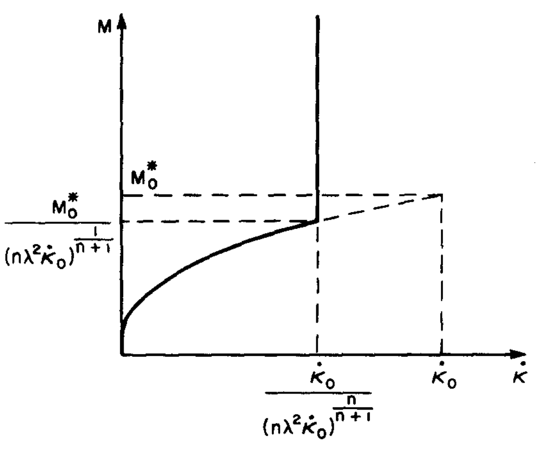


FIG. 4.

equations (46) and (57) give

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \left(\frac{M}{M_0^*}\right)^n \quad \text{for} \quad |M| \leq (M_0^*)^{\frac{n+1+\alpha}{n+1}} \tag{58a}$$

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \langle M \rangle \{|M|\}^{\frac{n\alpha}{n+1+\alpha}} \quad \text{for} \quad |M| \geq (M_0^*)^{\frac{n+1+\alpha}{n+1}}. \tag{58b}$$

The analogue of the complementary energy density, $\Omega(M) = \int \dot{\kappa}\, \mathrm{d}M$, is then

$$\Omega(M) = \frac{\dot{\kappa}_0^*}{n+1} \frac{M^{n+1}}{(M_0^*)^n} \quad \text{for} \quad |M| \leq (M_0^*)^{\frac{n+1+\alpha}{n+1}} \tag{59a}$$

$$\Omega(M) = \frac{(n+1+\alpha)\dot{\kappa}_0^*}{(n+1)(1+\alpha)}\{|M|\}^{\frac{(n+1)(1+\alpha)}{n+1+\alpha}} - \frac{n\dot{\kappa}_0^*}{(n+1)(1+\alpha)}(M_0^*)^{1+\alpha} \quad \text{for} \quad |M| \geq (M_0^*)^{\frac{n+1+\alpha}{n+1}}. \tag{59b}$$

The bending moment distribution associated with the optimal design can now be found by finding the statically admissible field $M(s)$ which minimizes

$$\int_S \Omega(M)\, \mathrm{d}s.$$

This can be done without previously determining $\dot{\kappa}_0^*$. We may then determine $\dot{\kappa}_0^*$ from the compliance constraint: using equations (58),

$$D = \mathbf{P} \cdot \mathbf{v} = \int_S M\dot{\kappa}\, \mathrm{d}s$$

$$= \int_{S_1} \dot{\kappa}_0^* \frac{M^{n+1}}{(M_0^*)^n} \mathrm{d}s + \int_{S_2} \dot{\kappa}_0^* (M)^{\frac{(n+1)(1+\alpha)}{n+1+\alpha}} \mathrm{d}s \tag{60}$$

$S_1$ and $S_2$ are determined from $M(s)$ by means of the constraints given in equation (58).

## 7. EXAMPLE

As a simple illustration of the procedure outlined in the previous section consider the optimal design of a beam (Fig. 5) which is fixed at both ends and carries a point load $P$ at the center. We require that the central displacement rate should be $\dot{\delta}$, so that

$$D = P\dot{\delta}. \tag{61}$$

Let us assume that the cost function is linear, i.e. $\alpha = 0$. Equations (58) then become

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \left(\frac{M}{M_0^*}\right)^n \quad \text{for} \quad |M| \leq M_0^* \tag{62a}$$

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \langle M \rangle \quad \text{for} \quad |M| \geq M_0^*. \tag{62b}$$

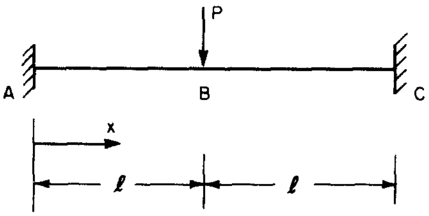


FIG. 5.

This relation is of the form shown in Fig. 4, with $n\lambda^2\kappa_0 = 1$ and $\dot{\kappa}_0 = \dot{\kappa}_0^*$. Further, $\Omega(M)$ becomes

$$\Omega(M) = \frac{\dot{\kappa}_0^*}{n+1}\frac{M^{n+1}}{(M_0^*)^n} \quad \text{for} \quad |M| \leq M_0^* \tag{63a}$$

$$\Omega(M) = \dot{\kappa}_0^*|M| - \frac{n}{n+1}M_0^*\dot{\kappa}_0^* \quad \text{for} \quad |M| \geq M_0^*. \tag{63b}$$

The symmetric fixed end beam under discussion has one indeterminacy. It is convenient to draw the bending moment distribution $M(x)$ for half the beam, as shown in Fig. 6. We let $M(a) = 0$, so that $|M(0)| = Pa/2$ and $|M(l)| = P(l-a)/2$. Further, since the slope of the linear function $M(x)$ is independent of $a$, we let $|M(a-\rho)| = |M(a+\rho)| = M_0^*$. Thus $S_1$ is given by $a-\rho \leq x \leq a+\rho$, and $S_2$ is given by $0 \leq x \leq a-\rho$ and $a+\rho \leq x \leq l$. By simple proportion

$$\rho = \frac{2M_0^*}{P}. \tag{64}$$

We assume for the present that $\rho < a$ and $\rho < l-a$, and determine $a$ by minimizing the total complementary energy. The total complementary energy is obtained by straightforward integration, and is given by

$$\frac{1}{M_0^*\dot{\kappa}_0^*}\int_S \Omega(M)\,ds = \left\{\frac{Pa^2}{2M_0^*} - \frac{2M_0^*}{P}\right\} + \left\{\frac{8M_0^*}{(n+2)(n+1)P}\right\} + \left\{\frac{P(l-a)^2}{2M_0^*} - \frac{2M_0^*}{P}\right\} \tag{65}$$

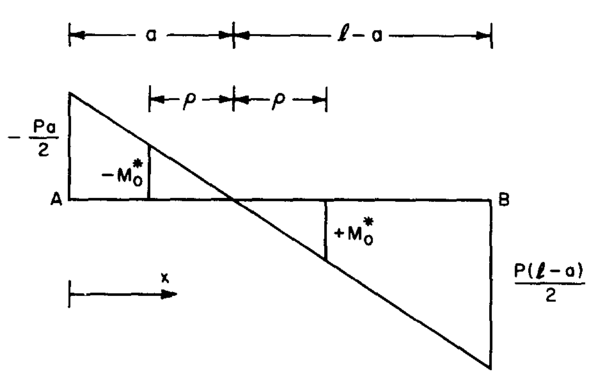


FIG. 6.

where the three terms are respectively the contributions from the regions $0 \le x \le a-\rho$, $a-\rho \le x \le a+\rho$ and $a+\rho \le x \le l$. Differentiating (65) with respect to $a$ and equating to zero, we find the result expected as a result of symmetry.

$$a = l/2. \tag{66}$$

We now note that if

$$\frac{2M_0^*}{P} = \rho = \frac{l}{2} \tag{67}$$

the region $S_1$ covers the whole beam and the optimum design is given by the minimum cross-section. From equation (55), $M_0$ is given by

$$M_0 = |M| \quad \text{for} \quad |M| \ge M_0^*. \tag{68}$$

The optimum design is shown in Fig. 7.

It remains now to translate this design into meaningful terms. We first determine $\dot{\kappa}_0^*$ from the compliance constraint, using equation (60). A further straightforward integration gives

$$\dot{\kappa}_0^* = \frac{P\dot{\delta}}{\dfrac{8}{n+2}\dfrac{(M_0^*)^2}{P} + M_0^* l\left(\dfrac{Pl}{4M_0^*} - \dfrac{4M_0^*}{Pl}\right)} = \frac{4(\dot{\delta}/l^2)}{1 - \dfrac{n}{n+2}\left(\dfrac{4M_0^*}{Pl}\right)^2}. \tag{69}$$

When $\rho = l/2, Pl/4M_0^* = 1$ and $\dot{\kappa}_0^* = 2(n+2)\dot{\delta}/l^2$. This is the limiting value of $\dot{\kappa}_0^*$ when the optimal design gives the minimum strength at each cross-section with the compliance constraint satisfied. For $Pl/4M_0^* < 1$, the minimum strength $M_0^*$ is too great to allow the compliance constraint to be satisfied, and the central displacement rate will be less than $\dot{\delta}$.

Suppose that the beam is of sandwich construction, with equal face plates of width $b$ and thickness $t$ separated by a distance $2d$. The core is assumed to carry shear forces. The face plate material is assumed to have a stress–strain rate relation of the form

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} = \left(\frac{\sigma}{\sigma_0}\right)^n \tag{70}$$

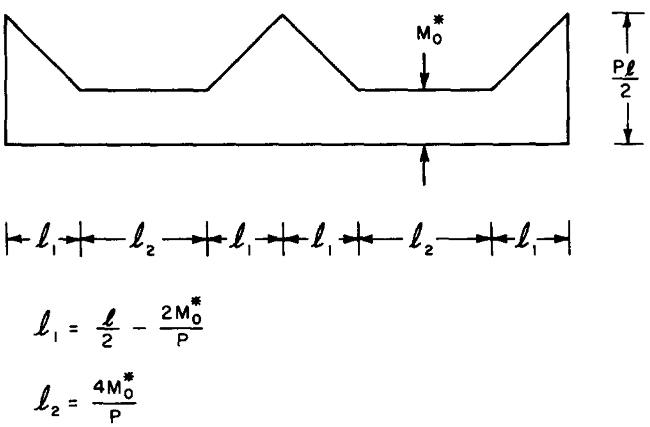


FIG. 7.

where $\sigma$, $\dot{\varepsilon}$ are stress and strain rate, and $\dot{\varepsilon}_0$ and $\sigma_0$ are constants with the dimensions of stress and strain rate. The moment–curvature relation for this beam will be

$$\frac{d\dot{\kappa}}{\dot{\varepsilon}_0} = \left(\frac{M}{2dbt\sigma_0}\right)^n. \tag{71}$$

This equation is rewritten as

$$\frac{\dot{\kappa}}{\dot{\kappa}_0^*} = \left[\frac{M}{(\dot{\kappa}_0^* d/\dot{\varepsilon}_0)^{1/n} 2dbt\sigma_0}\right]^n. \tag{72}$$

Thus

$$M_0 = 2dbt\sigma_0\left(\frac{\dot{\kappa}_0^* d}{\dot{\varepsilon}_0}\right)^{1/n}. \tag{73}$$

A linear relation between cost and $M_0$ can be achieved if it is assumed that $b$ and $d$ are constant, with $t$ variable and cost per unit length directly proportional to the area of the face plates $2bt$. The minimum plate thickness $t^*$ is used in conjunction with equation (69) to determine $M_0^*$. Eliminating $\dot{\kappa}_0^*$ from equations (69) and (73), $M_0^*$ is obtained from the solution of the equation

$$\left(\frac{M_0^*}{2dbt^*\sigma_0}\right)^n = \frac{d}{\dot{\varepsilon}_0}\left\{\frac{4(\dot{\delta}/l^2)}{1-\dfrac{n}{n+2}\left(\dfrac{4M_0^*}{Pl}\right)^2}\right\} \tag{74}$$

Once $M_0^*$ is determined, equations (69) and (73) are used to give a relation between $t(x)$ and $M_0(x)$;

$$t = \frac{M_0}{2db\sigma_0}\left\{\frac{1-\dfrac{n}{n+2}\left(\dfrac{4M_0^*}{Pl}\right)^2}{4d\dot{\delta}/\dot{\varepsilon}_0 l^2}\right\}^{1/n} \tag{75}$$

In the case of a linear elastic material, the velocity constraint $\dot{\delta}$ is replaced by a displacement constraint $\delta$. The constitutive relation is modified by putting $\dot{\varepsilon}_0 = 1$, $\sigma_0 = E$, $n = 1$ and interpreting $\dot{\varepsilon}$ as a strain rather than a strain rate.

The rigid–plastic case ($n \to \infty$) differs from the case for finite $n$ in that the computation of $\dot{\kappa}_0^*$ is unnecessary. We proceed directly from equation (68) to equation (72). With $n \to \infty$, equation (72) is interpreted to give

$$M_0 = 2bdt\sigma_0 \tag{76a}$$

with

$$M_0^* = 2bdt^*\sigma_0 \tag{76b}$$

## 8. SUMMARY

The optimal design technique presented in this paper in essence replaces the problem of minimizing a cost function subjected to certain constraints by one in which a "complementary energy" function must be minimized. This latter problem is the standard structural analysis problem; the "complementary energy" is a function of the class of statically admissible bending moments for the structure, and can be expressed in terms of independent parameters whose number is equal to the degree of indeterminacy.

In practical terms the utility of this technique depends solely on whether the resulting calculations are easier than those required by other methods. This aspect of the technique has not yet been studied for a sufficiently wide range of practical structures. It is clear, however, that the method will be of great advantage when the structural configuration is complex but the degree of indeterminacy is small.

The method is readily generalized to more complex structural types (e.g. bar structures in which more than one generalized strain contributed to the deformation, and to plates and shells). Further, as will be shown in future papers, it can be generalized to deal with piecewise uniform cross-sections and multi-purpose loading with compliance constraints.

*Acknowledgment*—The author is indebted to Professor W. Prager for his helpful advice and criticism.

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(*Received* 22 *October* 1969; *revised* 23 *February* 1970)

**Абстракт**—Рассматривается оптимальный расчет конструкций, изготовленных из материжла, удовлетворяющего степенному закону вязкоупругости. Конструкция должна быть расчитана на минимум веса, при условии определенной скорости действия внешних нагрузок. Оказывается, что задачу оптимального расчета можно рассматривать в смысле задачи расчета конструкции с ковститутивной зависимостью, модофицировнной условиями необходимыми и достаточными для установления оптимализации. На основе предложенной обработки можно востановить линейные упруше и жестко-пластические задачи.